STABLE BUNDLES AS FROBENIUS MORPHISM DIRECT IMAGE

CONGJUN LIU AND MINGSHUO ZHOU

Abstract. Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$, and let $F : X \to X_1$ be the relative Frobenius map. We show that a vector bundle E on X_1 is the direct image of some stable bundle X if and only if instability of F^*E is equal to $(p-1)(2g-2)$.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by $\mathcal{O}_X \to \mathcal{O}_X, f \mapsto f^p$. Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over k. One of the themes is to study its action on the geometric objects on X. Recall that a vector bundle E on a smooth projective curve is called semi-stable (resp. stable) if $\mu(E') \leq \mu(E)$ (resp. $\mu(E') < \mu(E)$) for any nontrivial proper subbundle $E' \subset E$, where $\mu(E)$ is the slope of E. It is known that F_* is preserves the stability of vector bundles (cf.[3]), but F^* does not preserve the semi-stability of vector bundle(cf.[1] for example).

Semi-stable bundles are basic constituents of vector bundles in the sense that any bundles E admits a unique filtration

$$\mathrm{HN}_\bullet(E) : 0 = \mathrm{HN}_0(E) \subset \mathrm{HN}_1(E) \subset \cdots \subset \mathrm{HN}_\ell(E) = E,$$

which is the so called Harder-Narasimhan filtration, such that

(1) $\mathrm{gr}_i^{\mathrm{HN}}(E) := \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$ $(1 \leq i \leq \ell)$ are semistable;
(2) $\mu(\mathrm{gr}_1^{\mathrm{HN}}(E)) > \mu(\mathrm{gr}_2^{\mathrm{HN}}(E)) > \cdots > \mu(\mathrm{gr}_\ell^{\mathrm{HN}}(E))$.

The rational number $\mathrm{I}(E) := \mu(\mathrm{gr}_1^{\mathrm{HN}}(E)) - \mu(\mathrm{gr}_\ell^{\mathrm{HN}}(E))$, which measures how far is a vector bundle from being semi-stable, is called the instability of E. It is clear that E is semi-stable if and only if $\mathrm{I}(E) = 0$.

Given a semi-stable bundle E on X_1, then F^*E may not be semi-stable, so it is natural to consider the instability $\mathrm{I}(F^*E)$. In [4, Theorem 3.1], the author prove $\mathrm{I}(F^*E) \leq (\ell-1)(2g-2)$, where ℓ is the

Date: November 29, 2012.

length of Harder-Narasimhan filtration of F^*E. If $E = F_*W$ where W is stable bundle on X, we know, by Sun's theorem ([3, theorem 2.2]), that E is stable, the length of Harder-Narasimhan filtration of F^*E is p and $\mathrm{I}(F^*E) = (p-1)(2g-2)$. Thus $\mathrm{I}(F^*E) = (p-1)(2g-2)$ is a necessary condition that E is a direct image under Frobenius. In this short note, we show the following theorem:

Theorem 1.1. *Let E be a stable vector bundle over X. Then the following statements are equivalent:*
(1) There exists a stable bundle W such that $E = F_*W$;
(2) $\mathrm{I}(F^*E) = (p-1)(2g-2)$.

The case $\mathrm{rk}E = p$ was proved in [2]. Our observation is that the arguments in [2] and Sun's theorem together imply the general case.

2. Proof of the theorem

Let X be a smooth projective curve over an algebraically closed field k with $\mathrm{char}(k) = p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by the homomorphism

$$\mathcal{O}_X \to \mathcal{O}_X, \qquad f \mapsto f^p$$

of rings. Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over k that satisfies

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[rr, bend left, "F_X"] \arrow[dr] & X_1 \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathrm{Spec}(k) \arrow[r, "F_k"] & \mathrm{Spec}(k) \end{tikzcd} .$$

For a vector bundle E on X,the slope of E is defined as

$$\mu(E) := \frac{\mathrm{deg}E}{\mathrm{rk}E}$$

where $\mathrm{rk}E$ (resp. deg E) denotes the rank (resp. degree) of E. Then

Definition 2.1. A vector bundle E on a X is called semi-stable (resp. stable) if for any nontrivial proper subbundle $E' \subset E$, we have

$$\mu(E') \leq (resp. <)\mu(E).$$

Theorem 2.2. *(Harder-Narasimhan filtration) For any vector bundle E, there is a unique filtration*

$$\mathrm{HN}_\bullet(E) : 0 = \mathrm{HN}_0(E) \subset \mathrm{HN}_1(E) \subset \cdots \subset \mathrm{HN}_\ell(E) = E,$$

which is the so called Harder-Narasimhan filtration, such that

(1) $\mathrm{gr}_i^{\mathrm{HN}}(E) := \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$ *($1 \le i \le \ell$) are semistable;*
(2) $\mu(\mathrm{gr}_1^{\mathrm{HN}}(E)) > \mu(\mathrm{gr}_2^{\mathrm{HN}}(E)) > \cdots > \mu(\mathrm{gr}_\ell^{\mathrm{HN}}(E))$.

By using this unique filtration of E, an invariant $\mathrm{I}(E)$ of E, which is called the instability of E was introduced (see [3] and [4]). It is a rational number and measures how far is E from being semi-stable.

Definition 2.3. Let $\mu_{\max}(E) = \mu(\mathrm{gr}_1^{\mathrm{HN}}(E))$, $\mu_{\min}(E) = \mu(\mathrm{gr}_\ell^{\mathrm{HN}}(E))$. Then the instability of E is defined to be

$$\mathrm{I}(E) := \mu_{\max}(E) - \mu_{\min}(E).$$

It is easy to see that a torsion free sheaf E is semi-stable if and only if $\mathrm{I}(E) = 0$.

For any semi-stable bundle E, let

$$\mathrm{HN}_\bullet(F^*E) : 0 = \mathrm{HN}_0(F^*E) \subset \mathrm{HN}_1(F^*E) \subset \cdots \subset \mathrm{HN}_\ell(F^*E) = F^*E$$

be the Harder-Narasimhan filtration of F^*E. Then we have the following lemma, which is implicit in [2].

Lemma 2.4. *For any semi-stable bundle E, we have*

$$\mu_{max}(F^*E) \le p \cdot \mu(E) + (p-1)(g-1);$$
$$\mu_{min}(F^*E) \ge p \cdot \mu(E) - (p-1)(g-1),$$

and if $\mathrm{I}(F^*E) = \mu_{\max}(F^*E) - \mu_{\min}(F^*E) = (p-1)(2g-2)$. *Then*

$$\mu_{max}(F^*E) = p \cdot \mu(E) + (p-1)(g-1);$$
$$\mu_{min}(F^*E) = p \cdot \mu(E) - (p-1)(g-1).$$

Now we prove our theorem by using this lemma and Sun's theorem on stability of Frobenius direct images.

Proof of Theorem 1.1. (1) $\Rightarrow$ (2) is contained in [3].
We prove (2) $\Rightarrow$ (1) here. Since $\mathrm{I}(F^*E) = (p-1)(2g-2)$, we have $\mu_{max}(F^*E) = p{\cdot}\mu(E) + (p-1)(g-1)$, $\mu_{min}(F^*E) = p{\cdot}\mu(E) - (p-1)(g-1)$ by lemma 2.4. We consider the surjection

$$F^*E \to gr_\ell^{HN}(F^*E).$$

The bundle $gr_\ell^{HN}(F^*E)$ is semi-stable of slope $\mu_{min}(F^*E)$. Replaced $gr_\ell^{HN}(F^*E)$ by a stable graded piece W in Jordan-Hölder filtration of $gr_\ell^{HN}(F^*E)$, we have a surjection

$$F^*E \to W,$$

where W is a stable bundle of slope $\mu(W) = \mu_{min}(F^*E) = p \cdot \mu(E) - (p-1)(g-1)$. By adjunction we have a non-trivial morphism

$$\psi : E \to F_*W.$$

By Sun's theorem (cf. [3, Theorem 2.2]), we know that F_*W is a stable bundle of slope

$$\mu(F_*W) = \frac{\mu(W)}{p} + \frac{(p-1)(g-1)}{p} = \mu(E).$$

Thus ψ induce an isomorphism:

$$E \cong F_*W.$$

$\square$

Acknowledgements: The authors would like to thank their advisor professor Xiaotao Sun for encouragements and many useful discussions.

References

[1] D.Gieseker: Stable vector bundles and the Frobenius morphism, Ann.Sci.École Norm. Sup. (4),**6**(1973),95-101.

[2] V. Mehta and C. Pauly : Semistability of Frobenius direct images over curves, Bull. Soc. Math. France, **135**(2007),105-117.

[3] X. Sun : Direct images of bundles under Frobenius morphism, Invent. math. **173**(2008),427-447.

[4] X. Sun : Remarks on semistability of G-bundles in Positive Characteristic, Comp. Math. **119**(1999),41-52.

Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing, P. R. of China

E-mail address: liucongjun@amss.ac.cn

Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing, P. R. of China

E-mail address: zhoumingshuo@amss.ac.cn